10/13/21

Last time:

$D(\vec{p})>0$ and $f_{xx}(\vec{p})>0 \Rightarrow \vec{p}$ a local min "up like a cup"

$D(\vec{p})>0$ and $f_{xx}(\vec{p})<0 \Rightarrow \vec{p}$ a local max "down like a frown"

$D(\vec{p})<0 \Rightarrow p$ is a saddle point

1. ★ if $D(\vec{p})=0$ can't conclude anything!
2. ★ could use f_{yy} instead of f_{xx}

• ex: Classify CP using 2nd der test

$f(x,y,z) = x^2 + xy + y^2 + y$

Critical pt if $\nabla f = 0$

$\nabla f = \langle 2x+y,\ x+2y+1 \rangle$

$\nabla f = 0$ iff $2x+y=0$, $x+2y+1=0$

$\begin{cases} y=-2x \\ x+2(-2x)+1=0 \end{cases} \Rightarrow \begin{cases} x=\frac{1}{3} \\ y=\frac{-2}{3} \end{cases}$

$3x+1=0$

$CP=\left(\frac{1}{3}, \frac{-2}{3}\right)$

• via second der test,

$f_{xx}\left(\frac{1}{3}, \frac{-2}{3}\right)=2 \quad f_{yy}\left(\frac{1}{3}, \frac{-2}{3}\right)=2 \quad f_{xy}=1$

$D_{x,y} = f_{xx} \cdot f_{yy} - f_{xy}^2 = 2 \cdot 2 - 1^2 = 3$

Classify pts: at $P=\left(\frac{1}{3}, \frac{-2}{3}\right)$, $D(P)=3>0$ & $f_{xx}\left(\frac{1}{3}, \frac{-2}{3}\right)=2>0$

→ a local minimum point

$f\left(\frac{1}{3}, \frac{-2}{3}\right) = x^2 + xy + y^2 + y$ @ $\left(\frac{1}{3}, \frac{-2}{3}\right) = \frac{1}{3}$

• ex: Classify critical points of $f(x,y) = x^3 + y^3 - 3x^2 - 3y^2 - 9y$

Critical points:

$\nabla f = \langle 3x^2 - 6x,\ 3y^2 - 6y - 9 \rangle$

$\nabla f = 0 = 3(x-2),\ (y-3)(y+1)$

$x=0$ $y=3$

$x=2$ $y=-1$ $\Rightarrow$ CP: $(0,3), (2,3), (0,-1), (2,-1)$

	$x=0$	$x=2$
$y=3$	$(0,3)$	$(2,3)$
$y=-1$	$(0,-1)$	$(2,-1)$

grid for visualization

• via second der test,

$f_{xx} = \langle 6x-6 \rangle \quad f_{yy} = \langle 6y-6 \rangle \quad f_{xy} = 0$

$D_{x,y} = f_{xx} \cdot f_{yy} - f_{xy}^2 = 6^2(x-1)(y-1)$

analyzing:

@ $(0,3)$: $D_{0,3}<0$ saddle point

@ $(0,-1)$: $D_{0,-1}>0$ $f_{xx}(0,-1)<0$ so a local max $f(0,-1)=13$

@ $(2,3)$: $D_{2,3}>0$ $f_{xx}(2,3)>0$ so local min at $f(2,3)=-3$

@ $(2,-1)$: $D_{2,-1}<0$ saddle point

★ calculate values at min/max not saddle points

• ex: $f(x,y) = xy + e^{-xy}$

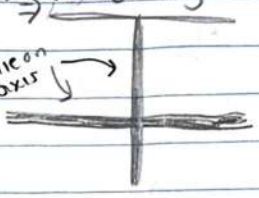
$\nabla f = \langle y - ye^{-xy}, x - xe^{-xy} \rangle$

$\nabla f = 0$ iff $y(1-e^{-xy}) = 0$ | $x(1-e^{-xy}) = 0$

$y=0$ or $1-e^{-xy}=0$ iff $e^{-xy}=e^0$ | $x=0$ or $1-e^{-xy}=0$

$\nabla f = 0$ iff $y=0 \;||\; (x=0 \;||\; y=0)$ $\Rightarrow x=0 \;||\; y=0$

$x=0 \;||\; (y=0 \;||\; x=0)$



$f_{xx} = y^2e^{-xy}$ $f_{yy} = x^2e^{-xy}$

$f_{xy} = (1-e^{xy})(1-xy)$

$D(x,y) = (xy)^2e^{-xy} - ((1-e^{-xy})(1-xy))^2$

$D(0,y) = 0 - ((0)(1-0))^2 = 0$ } when $D(x,y)$ we learn nothing.

$D(x,0) = 0$

?: Lagrange Multipliers

Goal: Build a method to systematically solve constrained optimization methods

Optimize $f(\vec{x})$ subject to $g_1(\vec{x}) = g_2(\vec{x}) = \dots g_N(\vec{x}) = 0$ ← constraints

• Observe: If we want extremes of F and want to live on level set $F(\vec{x}) = c$, we really want critical points of $F(\vec{x})$, because $\nabla F = \nabla c = 0$

$F(\vec{x}, \lambda_1, \dots, \lambda_N) = f(\vec{x}) - \lambda_1 g_1(\vec{x}) - \dots - \lambda_N g_N(\vec{x})$

because level set consideration (on website)

↳ the solutions to $f(\vec{x})$ occur only at CP of $F(\vec{x})$

↳ therefore, only need to solve $\nabla f = 0$ and find min/max values

• ex: Optimize $f(x,y) = xe^y$ where $x^2+y^2=2$

$x^2+y^2-2=0 = g(x,y)$

$F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$

$= xe^y - \lambda(x^2+y^2-2)$

$\nabla F = \langle e^y - 2\lambda x, xe^y - 2\lambda y, -(x^2+y^2-2) \rangle = 0$

$e^y - 2\lambda x = 0$ | $xe^y - 2\lambda y = 0$ | $x^2+y^2 = +2$

dividing

$y = x^2$

$\lambda = e^{y/x} = e^{1/2(1)} = \pm\frac{e}{2}$

Therefore $(1,1)$ is a possible extreme of F at $\pm e$ and $(-1,1)$

$x^2+x^4-2=0$

$(x^2+2)^2(x^2-1) = 0$

$x=1 \Rightarrow y=1$

$x=-1$

Solution:
$(-1,-1)$ max at $-e$
$(1,1)$ min at e